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On a super Lie group structure for the group of G^∞ diffeomorphisms of a compact G^∞ supermanifold

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Abstract

We show that the group of automorphisms, $D_G(M)$, of a compact Rogers' supermanifold, M, admits the structure of a G^∞ manifold. We establish that the space of paths on $D_G(M)$ based at the origin and the space of loops at the origin also admit G^∞ structures such that we obtain an exact sequence of G^∞ groups.

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0. Introduction

The work of Rogers [24] makes it possible to speak of compact supermanifolds. It is well known that the group of diffeomorphisms of a compact manifold is an infinite-dimensional Lie group. I. Singer asked me during a conversation several months ago if the group of automorphisms of a compact supermanifold admits the structure of a super Lie group.

We shall address this question in Section 2. In Section 1 we shall generalize Rogers' constructions so that in an infinite-dimensional context they are adequate for the strucutre that exists on the group of automorphisms of a compact Rogers' supermanifold. In Section 2 we observe that finite-dimensional supermanifolds can be considered as ordinary C^∞ manifolds with a specified G structure which we described. We then prove that the automorphisms of such a G structure on a compact supermanifold, $D_G(M)$, is a Lie subgroup of the group of C^∞ diffeomorphisms of the underlying C^∞ structure of the supermanifold in question. We proceed to construct explicitly a system of charts on $D_G(M)$ for

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which the chart changes are G^∞ morphisms, and show that the group operations are also G^∞.

In Section 3 we show that the space of paths on $D_G(M)$ admits in a canonical manner a supermanifold structure such that the evaluation map is a G^∞ map. Then we use a version of Lie's second theorem that we have proved elsewhere [15] to establish that the space of loops at the identity, $\Omega(D_G(M))$, is also a super Lie group so that we obtain a G^∞ exact sequence of super Lie groups

$$e \mapsto \Omega(D_G(M)) \mapsto C^\infty(I, D_G(M)) \mapsto \overset{\circ}{D}_G\ (M) \mapsto e,$$

where $\overset{\circ}{D}_G\ (M)$ is the connected component of the identity of $D_G(M)$.

1. Differential equations in infinite-dimensional graded topological vector spaces

We recall that a Hausdorff, sequentially complete, locally convex topological vector space V is called strongly bornological (resp. bornological) when any subset (resp. convex subset) absorbing all the bounded subsets of V is a neighborhood of the origin. In general, in this paper, the topological vector spaces which we deal will be strongly bornological. Note that metrisable locally convex topological vector spaces are strongly bornological as are countable inductive limits of strongly bornological spaces.

Let Γ be the Grassmannian ring of supernumbers generated by an arbitrary set $\mathcal{X} = \{x_i\}_{i\in I}$. We shall suppose that Γ has the locally convex topological vector space topology given by the inductive limit topology taken over its finitely generated subalgebras. With this topology Γ is a complete locally convex Z_2 graded commutative algebra (i.e. $ab = (-1)^{|a||b|}ba$, where $|a|$ designates the parity of a).

Definition 1.1. Let $E_1, \ldots, E_n, F$ be topological Z_2 modules over Γ_0, a continuous mapping

$$f : E_1 \times \cdots \times E_n \mapsto F$$

is a said to be an n-multimorphism when f is n-multilinear with repect to the ground field R and

$$f(e_1, \ldots, e_i\gamma, e_{i+1}, \ldots, e_n) = f(e_1, \ldots, e_i, \gamma e_{i+1}, \ldots, e_n), \qquad \gamma \in \Gamma_0$$

and

$$f(e_1, \ldots, e_n\gamma) = f(e_1, \ldots, e_n\gamma) = f(e_1, \ldots, e_n)\gamma, \qquad \gamma \in \Gamma_0.$$

We shall now describe the topological vector spaces upon which we shall model our supermanifolds.

Definition 1.2. Let M be a graded Cartesian product of Γ modules, $M = \prod \Gamma_i$, $i \in I$, $\Gamma_i = \Gamma\ \forall i$, and $(\prod \Gamma)_i = \prod(\Gamma_i)$, $i = 0, 1$. A Γ_0 submodule N of M will be called *typal*.

Definition 1.3. Give typal Γ_0 modules $Q_1, Q_2, Q_k \subset \prod_{i\in I_k} \Gamma_i$, a Γ_0 homomorphism $f : Q_1 \mapsto Q_2$ will be called *regular* when Γ is a subalgebra of an infinitely generated Grassmannian algebra Λ such that f extends to a Λ_0 homomorphism of $\tilde{Q}_1 = \Lambda_0 Q_1$ into $\tilde{Q}_2 = \Lambda_0 Q_2$, where $\tilde{Q}_k$ is considered as a submodule of the corresponding Λ_0*–module* $\prod_{i\in I_k} \Lambda_i$, $\Lambda_i \equiv \Lambda\ \forall i$.

Remark. Note that typicality and regularity are purely algebraic notions.

The notion of differentiability that we shall use throughout the sequel is as follows:

Definition 1.4. Given a Grassmannian algebra Γ, let V and W be typal topological modules over Γ_0 such that the underlying topologies are those of complete Hausdorff strongly bornological topological vector spaces, $U \subseteq V$ open. A function $f : U \mapsto W$ will be called super C^n or G^n when $\exists$ continuous k-multimorphisms in the k-terminal variables such that for $x \in U$ fixed, $D_x f(x; \star)$ is a regular mapping from V to W, and $D^k f(x; \cdots) : U \times V \times \cdots \times V \mapsto W$ for $k \leq n$ is such that

$$F_k(h) = f(x+h) - f(x) - 1/1!Df(x,h) - \cdots - 1/k!D^k f(x,h,\ldots,h), \qquad 1 \leq k \leq n,$$

satisfies the property that

$$G_k(t,h) = \begin{cases} F_k(th)/t^k, & \text{if } t \neq 0, \\ 0, & t = 0, \end{cases}$$

is continuous at $(0,h)\forall h \in V$.

We shall refer to $D_x f(x,\cdot)$ as the *Frechet* derivative.

Remark. When Γ is finite-dimensional and N is of the form $N = \prod_{i\in x_1} (\Gamma_1)_i \times \prod_{i\in x_0} (\Gamma_0)_i$, where $X_0, X_1 \subseteq X$, X a finite indexing set for the free module $M = \prod_{i\in X} \Gamma_i$, $\Gamma_i = \Gamma$ $\forall$i $\in X$, we obtain Rogers' definition of G^n.

Now we turn our attention to linear differential equations of the form

$$(*) \qquad y' = A(t)y, \quad t \in I \quad I = [0,1],$$

where $A(t)$ is a regular morphism of a Γ_0 module.

Definition 1.5. Consider the differential equation

$$y'(t) = A(t)y, \quad t \in I, \quad I = [0,1],$$

where $A(t)$ is a family of continuous linear operators on a bornological space E. The family of operators $\{A(t)\}$ will be called *tame* when

(i) $(t,x) \to A(t)x$ is C^∞;

(ii) $\{A(t)\}$ is a strongly bounded family of operators; that is, given a bounded set $B \subset E$, we have that $\bigcup_{t\in I} A(t)B$ is bounded in E;

(iii) given any bounded disk $B \subset E$ there exists a sequence of bounded disks $B_1, \ldots, B_n, \ldots$ so that:

(a) $\bigcup_{t\in I} A(t)B \subset B_1, \bigcup_{t\in I} A(t)B_n \subset B_{n+1}$;

(b) There exists a bounded disk D such that $B_n \subset E_D$ for all n and $F_n = \sum_{q\geq n}(1/q!)$ B_q converges to 0 in $E_D = \bigcup_{\lambda>0} \lambda D$ as n tends to infinity.

Proposition 1.1. *If $y' = A(t)y$ is a tame linear differential equation, then there exits a unique flow $y_A(t,x)$ such that $y'_A(t,x) = A(t)y_A(t,x)$ with $y_A(0,x) = x$; further, if $\{A(t)\}$ is a family of regular homomorphisms, then $\alpha \mapsto y_A(t,\alpha)$ is a continuous regular Γ_0 ismorphism for all $t \in I$, whose continuous inverse $\beta \mapsto g_A(t,\beta)$ is a smooth function in t satisfying $g'_A(t,x) = -g_A(t, A(t)x)$.*

It is useful to recall the form of the solution of $y'_A = A(t)y$, $y_A(0,x) = x$:

$$y_A(t,x) = x + \int_0^t A(s_1)x\,\mathrm{d}s_1 + \int_0^t A(s_1)\left\{\int_0^{s_1} A(s_2)(x)\,\mathrm{d}s_2\right\}\mathrm{d}s_1 + \cdots$$

The proof is essentially the same as for the analogous propositions in [16].

Definition 1.6. $A(t,s) \equiv A_s(t) : E \mapsto E, \lambda \in R$, called an *amenable* family of tame operators when

(a) $A : I \times I \times E \mapsto E$ is C^∞;

(b) Given any bounded disk $B \subset E$ there exists a sequence of bounded disks $B_1, \ldots, B_n, \ldots$ so that $\bigcup_{(t,s)\in I\times I} A(t,s)B \subset B_1, \bigcup_{(t,s)\in I\times I} A(t,s)B_n \subset B_{n+1}$;

(c) There exists a bounded disk D such that $B_n \subset E_D$ for all n and $F_n = \sum_{q\geq n}(1/q!)B_q$ converges to 0 in $E_D = \bigcup_{\lambda\geq 0} \lambda D$ as n tends to infinity.

We have proved in [16]:

Theorem 1.1. *Set $F_A(x,y)\zeta = y_A(x, g_A(y,\zeta))$, where $A(t,s)$ is an amenable family of tame operators. Then $\partial y_A/\partial\sigma(t,\sigma)\zeta = \int_0^t F_A(t,s)\partial A(s,\sigma)/\partial\sigma F_A(s,0)\zeta\,\mathrm{d}s$.*

Given a bornological locally convex topological vector space E, a collection, $\mathcal{B}$, is called a system of generators of the bounded sets of E when:

(a) given any bounded subset B of E, $\exists$ an element $C \in \mathcal{B}$ and $r > 0$ such that $B \subset rC$;

(b) $B_1, B_2 \in \mathcal{B} \Longrightarrow \exists B \in \mathcal{B}$ such tht $B_1 \cup B_2 \subset B$.

In what follows we shall use the C^∞ topology on function spaces where the range, E, is a bornological topological vector space, what we intend by the C^∞ topology we shall now describe.

Consider the space of C^∞ mappings from the unit interval to E, $C^\infty(I,E)$. We put a topology on $C^\infty(I,E)$ as follows: Let $\mathcal{C}$ be the set of closed bounded disks of E, N the set of nonnegative integers, and let $F(N,\mathcal{C})$ be the set of functions from N to $\mathcal{C}$; now, we define an order on $F(N, \mathcal{C})$ by $f < g$ when $f(i) \subseteq g(i)$ for every $i \in N$. With this order $F(N, \mathcal{C})$ becomes a directed set. For $\gamma \in F(N, \mathcal{C})$ let

$$\Gamma_\gamma = \{f \in C^\infty(I, E) \colon D^l f(x) \subseteq \gamma(l)\},$$

where $x \in I$, and let $C^\infty(I, E)_{\Gamma_\gamma} = \bigcup_{\lambda \geq 0} \lambda \Gamma_\gamma$ be the Banach space with norm $\|f\|_\Gamma = \inf\{\lambda \geq 0 \colon f \in \lambda \Gamma\}$. We provide $C^\infty(I, E) = \inf \lim_\gamma C^\infty(I, E)_{\Gamma_\gamma}$ with the locally convex inductive limit topology.

With the aid of Theorem 1.1 we are able to prove by essentially the same proof as in [16] the following:

Theorem 1.2. *Let $T_1, \ldots, T_n$ and E by typal Γ_0 modules whose underlying topological vector space structures are strongly bornological and in which bounded sets are relatively compact. Suppose that $U \subseteq E$ is open, $I = [0, 1]$, and $F : I \times U \times T_1 \times \cdots \times T_n \mapsto E$ is a G^∞ function which is a multiaffine morphism in $T_1, \ldots, T_n$; that is, such that for $i = 1, \ldots, n$, $t_i \mapsto F(t, u, t_1, \ldots, t_i, \ldots, t_n)$ is of the form $L_i(t_1, \ldots, t_n) + c_i(t, u, t_1, \ldots, \hat{t}_i, \ldots, t_n)$, where L_i and c_i are G^∞ functions such that L_i is an n-multimorphism and c_i does not depend on t_i. Given $x_0 \in U$, $t_0 \in (0, 1)$, $t_i^0 \in T_i$ let $\mathcal{B}$ (resp. $\mathcal{C}_i$) be a $U - \{x_0\}$ (resp. T_i) system of generators of the bounded sets in E (resp. T_i), suppose that for $B \in \mathcal{B}$, $C_i \in \mathcal{C}_i$, there exists a sequence $B_1, \ldots, B_n, \ldots$ of bounded disks in E satisfying $x_0 + B + (\delta/1!)B_1 + \cdots + (\delta^m/m!)B_m \subseteq U$ for all m, such that if $|t - t_0| \leq \delta$, we have that $F(t, B, y_1, \ldots, y_n) \subseteq B_1$, $\partial F/\partial E(t, w, y_1, \ldots, y_n)B_k \subseteq B_{k+1}$, $y_i \in C_i$, and $\sum_{q \geq n} (\delta^q/q!)B_q$ converges to 0 in E_l for some $l \in \mathcal{B}$ as n tends to infinity. Then there exist an open neighborhood $U_0 \subseteq U$ of x_0 and $V_i \subseteq T_i$ of t_i^0, and a unique function $\phi : [t_0 - \delta, t_0 + \delta] \times U_0 \times V_1 \times \cdots \times V_n \mapsto U$ satisfying $D_t\phi(t, s, t_1, \ldots, t_n) = F(t, \phi(t, s, t_1, \ldots, t_n), t_1, \ldots, t_n)$, $\phi(t_0, x, t_1, \ldots, t_n) = x$; further, $\Phi(x, t_1, \ldots, t_n) = \Phi(\star, x, t_1, \ldots, t_n)$ defines a C^∞ function $\Phi : U_0 \times V_1 \times \cdots \times V_n \mapsto C^\infty([t_0 - \delta, t_0 + \delta], E)$, where $C^\infty([t_0 - \delta, t_0 + \delta], E)$ has the C^∞ topology.*

2. Infinite-dimensional supermanifolds

A supermanifold in this paper will be a manifold modeled on a typal Γ_0 topological module, E, with G^∞ chart changes.

In the classical theory there is a one–one correspondence between derivations of the algebra of germs of smooth functions at a point and the equivalence classes of smooth curves through a point. For nontrivial Γ in the case of a supermanifold the space of superderivations of G^∞ functions is strictly larger than the classes of smooth curves through a point giving rise to a vector at a point. Given a supermanifold modeled on a typal Γ_0 module E, typically a vector would correspond uniquely to an element of E, while a superderivation

$$X(ab) = X(a)b + (-1)^{|a||X|} a X(b)$$

would correspond to an element of $\Gamma \otimes_{\Gamma_0} E$. A super Lie group, G, thus has two algebraic structures associated to it; one is its classical Lie algebra, $\mathcal{G}$, of right invariant vector fields with the classical Lie bracket satisfying Jacobi's identity, here $\mathcal{G}$ is a Γ_0 module such that the bracket satisfies $[\gamma_0 a, b] = \gamma_0[a, b]$. The left invariant superderivations, on the other hand, correspond to an extension by scalars of the smooth vector fields by Γ; that is, the super Lie algebra of left invariant superderivations is isomorphic to $\Gamma \otimes_{\Gamma_0} \mathcal{G}$. The superderivations satisfy

$$(**) \qquad (-1)^{|A||C|}[A, [B, C]] + (-1)^{|B||A|}[B, [C, A]] + (-1)^{|C||B|}[C, [A, B]] = 0,$$

$(**)$ is called the super Jacobi identity.

Suppose now that $E = (\Gamma_0)^m \times (\Gamma)^n$ is finite-dimensional, and let G be the group of continuous regular automorphisms of the Γ_0 structure of E. Let M be a supermanifold modeled on E, the supermanifold structure on M induces a G-structure on M; that is, reduces its underlying C^∞ tangent bundle to a G-bundle, since the Frechet derivatives of the transition functions are in G.

We shall call a C^∞ manifold, M, of dimension $2^{N-1}(m+n)$ modeled on E, a *quasi-supermanifold* of type (m, n), when the tangent bundle can be reduced to the group G. Now given the fact that a G^∞ mapping is a C^∞ mapping whose derivative is a regular map, we deduce that the automorphisms of the supermanifold structure of M are precisely the C^∞ diffeomorphisms of M which are also isomorphisms of its quasi-supermanifold structure; that is, it is the group of automorphisms of the associated G-structure of M.

Observe that the group $G \subseteq \mathrm{GL}(2^{N-1}(m+n), R)$ is locally convex; that is, there exits a fundamental system of convex neighborhood of the identity of $G \subseteq \mathrm{GL}(2^{N-1}(m+n), R)$ with respect to the canonical vector space structure on the $(2^{N-1}(m+n))^2$-dimensional space of square matrices. Note, firstly, that $\mathrm{GL}(q, R)$ is always locally convex, since $I + A$ is invertible for $\|A\| < 1$, and observe that for $\rho \in \Lambda_N$, $G_\rho = \{A \in \mathrm{GL}(2^{N-1}(m+n), R) \colon A\rho = \rho A\}$ is locally convex; further, the property or regularity is a locally convex property, thus G is locally convex.

It is known [14] that when $G \subset \mathrm{GL}(n, R)$ is locally convex then the group of automorphisms of a G-structure on a compact C^∞ manifold M is a Lie subgroup of the group of C^∞ diffeomorphisms of M. One of the purposes of this note is to show that the group of automorphisms of the supermanifold structure on M, $D_G(M)$, is itself an infinite-dimensional super Lie group with respect to the graded algebra Γ.

In what follows we shall suppose that M is a compact connected G^∞ supermanifold. The supermanifold structure on M determines canonically a typal Γ_0 module structure on T_xM for each $x \in M$. The elements $\phi \in D_G(M)$ determine regular Γ_0-homomorphisms

$$T\phi : T_xM \mapsto T_{\phi(x)}M,$$

which implies that the Lie subalgebra of the C^∞ vector fields on M corresponding to $D_G(M)$ considered as a C^∞ Lie subgroup of $\mathrm{Diff}^\infty(M)$ consists precisely of the G^∞ vector fields on M with respect to the canonically determined supermanifold structure on TM; we shall designate this Lie algebra by $\mathcal{S}$.

The definitions imply that:

Proposition 2.1. *$\mathcal{S}$ is a Γ_0-module such that $[a\lambda, b] = [a, \lambda b]$ and $[a, b\lambda] = [a, b]\lambda$ for $\lambda \in \Gamma_0$.*

We are now in a position to prove the following theorem.

Theorem 2.1. *With respect to the Z_2 graded algebra Γ, $D_G(M)$ is a super Lie group.*

Proof. We choose charts on $D_G(M)$ that are given by means of the exponential, $\exp_G$, of a G-connection; for details see [14]. We then obtain a chart at the identity $\tilde{\exp} : \mathcal{S} \mapsto D_G(M)$ defined by $\tilde{\exp}_G(\alpha)(x) = \exp_G(\alpha(x))$ for α sufficiently G^∞ small.

Using normal coordinates and calculating we find

$$D_x(\tilde{\exp}_G(X))(x, \alpha) = (x + X_x, \alpha + g_x(\alpha)),$$

where g_x is in the image of the canonical representation of the Lie algebra of G in the Lie algebra of the endomorphisms of T_xM given by the G-structure on M, which, since G is locally convex, consists of regular Γ_0 homomorphisms, which implies that the differentials of the changes of charts given by $\tilde{\exp}$ are $G^\infty(M, \Gamma)$-homomorphisms. Given $f \in D_G(M)$, define $\mathcal{S}_f = \{g : M \mapsto TM : \pi \circ g = f$, where $\pi : TM \mapsto M$ is the canonical projection$\}$ we construct a chart at f by $\tilde{\exp} : \mathcal{S}_f \mapsto D_G(M)$ as above. With these charts which determine a G^∞ structure on $D_G(M)$, right multiplication by g in $D_G(M)$ is represented at the infinitesimal level simply by composition from the right $\mathcal{S}_f \mapsto \mathcal{S}_{f\circ g}$, $\alpha \mapsto \alpha \circ g$. This is clearly a G^∞ mapping. Left multiplication by f in $D_G(M)$ at the infinitesimal level is represented by $\Xi : \mathcal{S}_g \mapsto \mathcal{S}_{f\circ g}$, where $\Xi(\alpha)(x) = T_{g(x)}f(\alpha(x))$. As T_xf is a regular homomorphism for all $x \in M$ it follows that right multiplication is indeed a G^∞ mapping. As right and left multiplication are G^∞, to see that inversion is everywhere G^∞ it suffices to observe that inversion is G^∞ at the identity. Since the Frechet derivative of inversion at the identity is the multiplication by -1, we conclude that inversion is an everywhere G^∞ mapping. □

3. Space of paths

The C^∞ topology on $C^\infty(I, \mathcal{S})$ is the underlying topology of an infinite-dimensional Lie group structure. To describe this Lie group, we first recall (see [16]) that a topological Lie algebra $\mathcal{E}$ is called *preintegrable* when given any closed bounded disk $B \subseteq \mathcal{E}$ there exists a sequence of closed bounded disks $B_1, \ldots, B_n, \ldots$ with

(i) $\mathrm{ad}_B(B) = \bigcup_{b_1, b_2 \in B}[b_1, b_2] \subseteq B_1$,
(ii) $\mathrm{ad}_B(B_n) \subseteq B_{n+1}$,
(iii) $\sum_{q\geq n}(1/q!)B_q$ converages to 0 in $\mathcal{E}_C = \bigcup_{\lambda\geq 0}\lambda C$ for some bounded closed disk C as q tends to ∞. In [16] we show:

Proposition 3.1. *Let $\mathcal{G}$ be a preintegrable topological Lie algebra, and $v : I \mapsto \mathcal{G}$ a C^∞ function. Then there exists a unique flow $\phi_v(t, x)$ of the differential equation $y' = [v(t), y]$.*

Proposition 3.1 puts us in position to define a product on $C^\infty(I, \mathcal{S})$:

$$(***) \qquad (v \star w) = v(t) + \phi_v(t, w(t)).$$

The Frechet derivative of left multiplication is given by $D_x L_v(x; \alpha)(t) = \phi_x(t, \alpha(t))$; and the Frechet derivative of right multiplication is given by $D_x R_w(x; \beta)(t) = \beta(t) + \int_0^t F_x(t,s)[w(s), F_x(s,0)\beta(t)]\,ds$, where $F_x(t,s) = \phi_x(t, \psi_x(s,\cdot))$, $\psi_x(s,\cdot)$ being the inverse of $\phi_x(s,\cdot)$. Now defining the Γ_0 action on $C^\infty(I, \mathcal{G})$ by $(\gamma\alpha)(t) = \gamma\alpha(t)$, we obtain that the Γ_0 bilinearity and regularity of the bracket imply that multiplication in this group structure is G^∞. To see this we utilize the infinite series expression of the solution of a linear differential equation in bornological spaces (cf. Section 1).

In [16] we showed that inversion is given by $g^{-1} = y_g$, where y_g is equal to the solution of a differential equation smoothly parametrized by g: $y' = F(t, y, g) = \psi_g(t, -g'(t) - [g, \phi_g(t, y)])$. Again the Γ_0 bilinearity of the bracket implies that $y \mapsto y^{-1}$ is a G^∞ on $C^\infty(I, \mathcal{S})$.

In [15] we defined a Lie group G as *nice* when the following conditions are satisfied: Given a Lie group G modeled on a complete bornological space $\mathcal{G}$ the manifold structure on G gives a local trivialization of the tangent bundle TG at $e \in G$ over a coordinate neighborhood of G at e say (U, ϕ), where $\phi(e) = O \in \mathcal{G}$. Identity $(T|U)_e$ with $\phi(U) \times \mathcal{G}$ by means of ϕ. Now, in general, a right invariant vector field ζ over U with respect to this trivialization will define a nonconstant C^∞ function $X_\zeta : U \mapsto \mathcal{G}$. We say that the Lie group G is nice when there is a $\phi(U)$-system of generators of the bounded sets of $\mathcal{G}$, $\mathcal{B}$, so that given any $B \in \mathcal{B}$ and any closed bounded $C \subset \mathcal{G}$ there exists a sequence of bounded sets in $\mathcal{G}$, C_n, $n \geq 1$, and $0 < \epsilon < 1$ such that

(1) $X_\zeta(\epsilon C) \subset C_1$ for $\zeta \in \epsilon B$;

(2) $DX_\zeta(\epsilon C + \epsilon/1!C_1 + \cdots + \epsilon^n/n!C_n; C_n) \subset C_{n+1}$ for $\zeta \in \epsilon B$;

(3) there exists a positive integer p and $D \in \mathcal{B}$ such that $D_n = \sum_{q \geq n}(\epsilon^q/q!)C_q \subset D$ for $n \geq p$ converges to 0 in $\mathcal{G}_D = \bigcup_{\lambda \geq 0} \lambda D$.

It follows from the definitions that a Lie subgroup of a nice Lie group is nice, which implies that $D_G(M)$ is a nice C^∞ Lie group. In [16] we showed that there exists a Lie group isomorphism, σ^{-1}, from the C^∞ Lie group structure defined above on $C^\infty(I, \mathcal{S})$ to the space of C^∞ paths at the identity, $C_0^\infty(I, D_G(M))$, where $\sigma(f)(t)$ is defined by logarithmic differentiation; that is, $\sigma(f)(t) = f'(t) \star f(t)^{-1}$, '$\star$' begin, by abuse of notation, the Frechet derivative of the right Lie group multiplication on $D_G(M)$ at $f(t)$ by $f(t)^{-1}$. Further, the endpoint evaluation Lie group homomorphism for the C^∞ topology on $C^\infty(I, D_g(M))$ $\mathrm{ev}: C_0^\infty(I, D_G(M)) \mapsto D_G(M)$ is given by $\mathrm{ev}((f))(1) = f(1) = y(1)$, where $y(t)$ is the solution to the equation $y'(t) = \sigma(f)(t) \star y(t)$, $y(0) = e$, which corresponds locally to an integral equation of the form: $y_v(t) = \int_0^t F(v, y_v(s))\,ds$, $y_v(0) = 0$; where $v \mapsto y_v$ is C^∞, $y_0(t) \equiv 0$, $F(0, y) = 0$, $F(v, 0) = v$. To show that ev is a super Lie group homomorphism it would suffice to show that the associated Lie algebra homomorphism $D_{v=0}(\mathrm{ev} \circ \sigma^{-1}) : C^\infty(I, \mathcal{S} \mapsto \mathcal{D}_G(M)$ is a regular Γ_0 homomorphism. The properties of $F(v, y)$ above imply that $D_{v=0}(\mathrm{ev} \circ \sigma^{-1}) = \int_0^1 : C^\infty(I, \mathcal{S}) \mapsto \mathcal{S}$; however, the bracket in $C^\infty(I, \mathcal{S})$ is defined by $[f, g](t) = D_t[\int_0^t f(s)\,ds, \int_0^t g(s)\,ds]$. The immediately preceding paragraph shows:

Theorem 3.1. *The evaluation map defines a G^{∞} homomorphism from the super Lie group of C^{∞} paths at the identity of $D_G(M)$ onto the connected component of the identity of $D_G(M)$.*

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